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ON THE PROBLEM OF THE ELASTIC STABILITY OF A LOCALLY LOADED CYLINDRICAL SHELL

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The influence of the nonlinear nature of axisymmetric deformation of a semi-infinite circular cylindrical shell compressed by a uniform transverse stress resultant at the endface, on its stability relative to infinitesimal nonsymmetric perturbations satisfying the hinge-support conditions on the endface, is studied.

An axisymmetric state of stress localized on a part of the length originates upon loading a circular cylindrical shell by a hoop load, upon heating it with a jump change in temperature on or between the supports, and under other effects concentrated over some cross section. Such a state can turn out to be unstable in the sense that as the load (temperature, etc.) reaches some critical level, the shell goes over into an adjacent equilibrium state with the formation of waves in the circumferential direction. It is known that an adjacent (convex) state, just as the initial (subcritical) state, is of a quite definite local nature (in the axial direction).

In the first theoretical investigations on the problem of stability of cylindrical shells under the conditions of a local axisymmetric state of stress, a classical formulation of this problem was used which does not take account of changes in the shape of the shell caused by subcritical strain [1 – 3]. The question of the need for a more general formulation of problems of the class considered was raised in [4], where the stability equations of a cylindrical shell heated between cold diaphragms were written taking account of subcritical curvature of its generator. The change in longitudinal curvature of the shell up to the time of buckling was determined by a linear formula. Later, solutions of a whole series of problems [5 – 10], which showed high sensitivity of the critical level of the external effects to a change in shell shape prior to buckling, were obtained in such a formulation.

Thus, values of the parameter τ in the formula for the critical temperature drop $\theta = \tau\mu^2 / \alpha$ ($\mu^2 = h / R \sqrt{3(1-\nu^2)}$, h and R are the shell thickness and radius, ν and α are the coefficients of transverse and temperature expansion), obtained taking account of subcritical bending in the stability problem of a cylindrical shell connected to a stiff cold diaphragm at an endface under uniform heating, are 40.5 [5], 39 [6], 34.6 [7] for different kinds of hinge support. The solution of the same problems in a classical formulation results in the values 8.74 [5], 8.56 [6], 8.85 [7]. For a step change in the temperature along the generator, the values found for τ were 30.2 [5], 32.5 [7], 33.9 [8], 34.6 [9], 29.8 [10] with subcritical bending taken into account, and 7.6 [3], 7.4 [5], 7.82 [7], 7.28 [8] without such bending. The effect caused by the subcritical curvature of the generator becomes fundamental in each of these problems and results in a 3.5 – 4.5-fold change in the critical value of the parameter τ.

Subcritical shell bending causes rotation of its generator at the connection with the diaphragm through an angle whose absolute value is $\tau\mu$. Therefore, for not too thin shells this angle becomes quite large prior to the time of buckling. Relative to such shells, doubt arises as to the admissibility of linearizing the expressions governing the subcritical bending strain components. In this connection, one of the characteristic problems of local cylindrical shell stability is considered below, where nonlinear equations describing the strain with arbitrary angles of rotation are used to determine the subcritical state. Some reasoning about the expediency of a nonlinear determination of the subcritical state has been expressed earlier in [11].

1. Let a semi-infinite circular cylindrical shell be loaded at the free endface by a uniform transverse stress resultant Q (Fig.1). Let M_α, N_α, ε_α, $\varkappa_\alpha$ $(\alpha = 1, 2)$, respectively, denote the specific bending moments, normal stress resultants, shear and bending strain components, where the subscript 1 will refer to the direction along the generator, and the subscript 2 to the circumferential direction. From the relationship

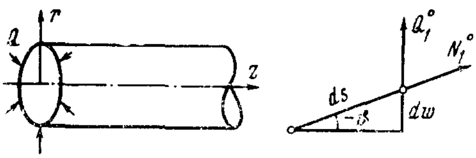

Fig. 1

$$dw^\circ / ds = -\sin\vartheta, \quad \varepsilon_2^\circ = w^\circ / R$$

where w° is the deflection and ϑ is the angle of rotation of the generator, follows the compatibility condition of the axisymmetric strains

$$R d\varepsilon_2^\circ / ds + \sin\vartheta = 0$$

Here s is a coordinate measured along the strained generator. The statistical and geometric characteristics of the axisymmetric subcritical state will be marked with a zero superscript. n conformity with their definition, we shall use the following exact expressions for the bending strain components

$$\varkappa_1^\circ = dv / ds, \quad \varkappa_2^\circ = (\cos\vartheta - 1) / R$$

Because of the absence of axial loads, the equilibrium equations of a shell element are

$$N_1^\circ = -Q_1^\circ \sin\vartheta, \quad N_2^\circ = R d Q_1^\circ / ds, \quad dM_1^\circ / ds - Q_1^\circ \cos\vartheta = 0$$

where $(Q_1^\circ$ is the radial stress resultant. By using the elasticity relationship

$$\varepsilon_2^\circ = A\,(N_2^\circ - \nu N_1^\circ), \quad M_1^\circ = D\,(\varkappa_1^\circ + \nu\varkappa_2^\circ) \quad (A = 1 / Eh, \quad D = Eh^3 / 12\,(1 - \nu^2))$$

we express the quantities ε_2° and M_1° in terms of the functions Q_1° and ϑ. The compatibility condition and the last of the equilibrium equations yield a nonlinear system for these functions. Using the notation

$$s = \sqrt{2CR}\,x, \quad Q_1^\circ = (D / CR)\,\eta(x), \quad C = h / \sqrt{12\,(1 - \nu^2)}, \quad \mu = \sqrt{2C / R}$$

let us write the nonlinear system as

$$\eta'' + \mu\nu\,(\eta\sin\vartheta)' + 2\sin\vartheta = 0, \quad \vartheta'' + \mu\nu\,(\cos\vartheta)' - 2\eta\cos\vartheta = 0$$

The primes here denote differentiation with respect to x. On the loaded edge there should be compliance with the conditions

$$Q_1^\circ = Q, \ M_1^\circ = 0$$

which become

$$\eta = \mu q, \ \vartheta' + \mu\nu(\cos\vartheta - 1) = 0 \qquad (x = 0), \ q = 2AQ / \mu^3$$

where q is a dimensionless loading parameter. Moreover, let us require boundedness of the functions η and ϑ at infinity. For thin shells $\mu\nu \ll 1$, which affords the possibility of formulating the boundary value problem posed in the simplified form

$$\begin{gathered}\eta'' + 2\sin\vartheta = 0, \ \vartheta'' - 2\eta\cos\vartheta = 0 \qquad (0 \leqslant x \leqslant \infty) \\ \eta = \mu q, \ \ \vartheta' = 0 \ \ (x = 0), \ \ \eta = \vartheta = 0 \ \ (x = \infty)\end{gathered} \tag{1.1}$$

Introducing the new variables

$$\varphi = \frac{\eta}{\mu}, \qquad \chi = \frac{\sin\vartheta}{\mu}, \qquad t = \int_0^x \cos\vartheta \, dx$$

we obtain in place of (1.1)

$$\begin{gathered}\varphi^{(2)} + 2\chi - \mu^2\chi(\varphi^{(2)}\chi + \varphi^{(1)}\chi^{(1)}) = 0, \quad \chi^{(2)} - 2\varphi = 0 \quad (0 \leqslant t \leqslant \infty) \\ \varphi = q, \ \ \chi^{(1)} = 0 \ \ (t = 0), \ \ \varphi = \chi = 0 \ \ (t = \infty)\end{gathered} \tag{1.2}$$

The superscript in parentheses here shows the order of the derivative with respect to t. The cubic nonlinearity of the dependent variables is isolated explicitly in such a writing. Eliminating the function φ from (1.2), we arrive at the following boundary value problem for the function χ:

$$\chi^{(4)} + 4\chi - \mu^2\chi(\chi^{(4)}\chi + \chi^{(3)}\chi^{(1)}) = 0 \qquad (0 \leqslant t \leqslant \infty) \tag{1.3}$$

$$\chi^{(2)} = 2q, \ \ \chi^{(1)} = 0 \ \ (t = 0), \ \ \chi^{(2)} = \chi = 0 \ \ (t = \infty) \tag{1.4}$$

Following [12], let us seek the solution of the problem as

$$\chi = a_1\zeta_1 + a_2\zeta_2 + \mu^2(a_3\zeta_1^3 + a_4\zeta_1^2\zeta_2 + a_5\zeta_1\zeta_2^2 + a_6\zeta_2^3) + \ldots \tag{1.5}$$

where ζ_1 and ζ_2 are linearly independent solutions of the equation $\zeta^{(4)} + 4\zeta = 0$, bounded at infinity, which have the form

$$\zeta_\alpha(t) = \exp(\lambda_\alpha t), \quad \lambda_1 = -1 + i, \quad \lambda_2 = -1 - i, \quad i = \sqrt{-1}$$

The coefficients $a_3, a_4, a_5, a_6, \ldots$ are determined successively from (1.3) in terms of the constants a_1 and a_2. In particular

$$a_3 = \frac{1}{40}a_1^3, \qquad a_4 = -\frac{1+3i}{20}a_1^2 a_2$$

$$a_5 = -\frac{1-3i}{20}a_1 a_2^2, \qquad a_6 = \frac{1}{40}a_2^3$$

The boundary conditions at $t = 0$ yield a nonlinear algebraic system to determine these constants

$$\lambda_1^2 a_1 + \lambda_2^2 a_2 + \mu^2 g_1(a_1, a_2) = 2q, \qquad \lambda_1 a_1 + \lambda_2 a_2 + \mu^2 g_2(a_1, a_2) = 0$$

$$g_1 = \frac{9}{40}\lambda_1^2 a_1^3 - \frac{13+9i}{10}a_1^2 a_2 - \frac{13-9i}{10}a_1 a_2^2 + \frac{9}{40}\lambda_2^2 a_2^3 + \ldots$$

$$g_2 = \frac{3}{40}\lambda_1 a_1^3 + \frac{3+4i}{10}a_1^2 a_2 + \frac{3-4i}{10}a_1 a_2^2 + \frac{3}{40}\lambda_2 a_2^3 + \ldots$$

Later we limit ourselves to the finite number of terms of the series (1.5) written down. We determine the constants a_1 and a_2 approximately from the equations

$$\lambda_1^2 a_1 + \lambda_2^2 a_2 = 2q - \mu^2 g_1(c_1, c_2), \quad \lambda_1 a_1 + \lambda_2 a_2 = -\mu^2 g_2(c_1, c_2)$$

where c_α is a solution of the linear system

$$\lambda_1^2 c_1 + \lambda_2^2 c_2 = 2q, \quad \lambda_1 c_1 + \lambda_2 c_2 = 0$$

For such a definition of the constants a_1 and a_2 the approximate solution obtained corresponds, in accuracy, to the first two members of the expansion of the solution of the boundary value problem (1.3),(1.4) in the small parameter μ^2. In this same approximation we have

$$t = \int_0^x \left(1 - \frac{\mu^2 \chi_0^2}{2}\right) dx, \qquad \chi_0 = \chi_\iota(x) = c_1 \zeta_1(x) + c_2 \zeta_2(x)$$

so that the variable t is a known function of x. After extracting real and imaginary parts of the introduced complex expressions, we obtain the functions needed later, which determine the circumferential stress resultant in the shell and the curvature of the generator of the subcritical state

$$\varphi^{(1)} = -qe^{-t}\left[(\alpha+\beta)\cos t - (\alpha-\beta)\sin t + p^2 e^{-2t} \times (27\sin 3t + 62\cos t + 34\sin t)\right] \quad (\alpha = 1 - 13p^2/2)$$

$$\chi^{(1)} = qe^{-t}\left[(\alpha-\beta)\cos t + (\alpha+\beta)\sin t + p^2 e^{-2t} \times (3\cos 3t - 14\cos t + 2\sin t)\right] \quad (\beta = 1 - 47p^2/2)$$

$$p^2 = \mu^2 q^2 / 40, \quad t = x - 5p^2\left[3 - e^{-2x}(2 + \cos 2x + \sin 2x)\right]$$

By formally discarding terms with the factoı μ^2 we obtain the solution of the linearized problem

$$\varphi_0' = -2qe^{-x}\cos x, \qquad \varphi_0 = qe^{-x}(\cos x - \sin x)$$

$$\chi_0' = 2qe^{-x}\sin x, \qquad \chi_0 = -qe^{-x}(\cos x + \sin x)$$

2. Taking account of the local nature of the buckling, the stability equations of the deformed cylindrical shell can be written in the form proposed by shallow shell theory [13]. Introducing the function v governing the increment of the normal stress resultants as

$$N_2 = -B\nabla_1^2 v, \qquad N_1 = -B\nabla_2^2 v \qquad (B = CEh)$$

and the function w governing the increment of the bending strains as

$$\varkappa_1 = -\nabla_1^2 w, \qquad \varkappa_2 = -\nabla_2^2 w$$

we arrive at the following description of the stability equations:

$$C\nabla^2\nabla^2 v + (R^{-1} + \varkappa_2^\circ)\nabla_1^2 w + \varkappa_1^\circ \nabla_2^2 w = 0$$

$$C\nabla^2\nabla^2 w - (R^{-1} + \varkappa_2^\circ)\nabla_1^2 v - \varkappa_1^\circ\nabla_2^2 v - B^{-1}(N_1^\circ \nabla_1^2 w + N_2^\circ \nabla_2^2 w) = 0$$

$$\left(\nabla_1^2 = \frac{\partial^2}{\partial s^2}, \ \nabla_2^2 = \frac{1}{R^2}\frac{\partial^2}{\partial y^2} + \frac{1}{R}\frac{dr}{ds}\frac{\partial}{\partial s}, \ \nabla^2 = \nabla_1^2 + \nabla_2^2\right)$$

Here $r(s)$ is the distance between the axis and a point on the strained shell surface, and y is an angular coordinate. Using the expressions

$$\frac{dr}{ds} = -\sin\vartheta, \qquad \frac{1}{R} + \varkappa_2^\circ = \frac{1}{R}\cos\vartheta, \qquad \varkappa_1^\circ = \frac{d\vartheta}{ds}$$

$$N_1^\circ = -\frac{B}{R}\eta\sin\vartheta, \qquad N_2^\circ = B\frac{d\eta}{ds}$$

going over to the variable x and assuming (n is an integer)

$$v = \sum_n v_n(x) \exp(iny), \qquad w = \sum_n w_n(x) \exp(iny)$$

we obtain a system of two ordinary differential equations for each value of n

$$P_n{}^2\vartheta_n + 2\cos\vartheta w_n'' - 2\mu^{-1}\vartheta' (\gamma^2 w_n + \mu \sin\vartheta w_n') = 0$$
$$P_n{}^2 w_n - 2\cos\vartheta v_n'' + 2\mu^{-1}\vartheta' (\gamma^2 v_n + \mu\sin\vartheta v_n') +$$
$$2\eta\sin\vartheta w_n'' + 2\mu^{-1}\eta' (\gamma^2 w_n + \mu\sin\vartheta w_n') = 0$$
$$\left(P_n = \frac{d^2}{dx^2} - \mu\sin\vartheta\frac{d}{dx} - \gamma^2, \quad \gamma = \mu n\right)$$

Introducing the functions φ, χ instead of the functions η, ϑ for the subcritical state, and taking account of the approximate nature of the solution of the nonlinear problem, we arrive at the final stability equations

$$P_n{}^2\vartheta_n + 2(1-\mu^2\chi_0^2) w_n'' - 2\chi^{(1)} (\gamma^2 w_n + \mu^2\chi_0 w_n') = 0$$
$$P_n{}^2 w_n - 2(1-\mu^2\chi_0^2) v_n'' + 2\chi^{(1)} (\gamma^2 v_n + \mu^2\chi_0 v_n') +$$
$$2\mu^2\varphi_0\chi_0 w_n'' + 2\varphi^{(1)} \sqrt{1-\mu^2\chi_0^2} (\gamma^2 w_n + \mu^2\chi_0 w_n') = 0$$

Let us be given the hinge-support boundary conditions for the perturbed state on the edge $x = 0$ (u_2 is the circumferential displacement)

$$N_1 = u_2 = w = M_1 = 0$$

The solution should vanish at infinity.

Therefore, we have a homogeneous boundary value problem dependent on the load parameter q. The least eigenvalue of this parameter defines the critical value of the external stress resultant Q. We hence call it the critical value.

Let us note that the solution of the formulated eigenvalue problem simultaneously yields the solution of the stability problem for a heated cylindrical shell hinge-connected to a cold stiff diaphragm at an endface. Such a shell buckles because of circumferential stresses caused by the limitation of the temperature strains in the reference section. The critical value of the parameter τ introduced above agrees with the critical value of the parameter q.

3. The problem of determining the critical value of the load parameter q was solved in finite differences by the matrix factorization method [14]. The computations were first carried out for $\mu = 0$, which corresponds to linearization of the equations of the subcritical state of the shell. A curve of the dependence of the first eigenvalue q_1 on the parameter γ, associated with the number of circumferential waves, is presented in Fig. 2 for this case. The curve has a minimum point for which $\gamma = \gamma^* \approx 4.9$, $q_1 = q^* \cong 32.7$. The length of the interval and the integration step were varied. Results have been obtained for an interval length 1.5π and a step $\pi / 120$, which assured calculation of the first eigenvalue q_1 to 0.5% accuracy.

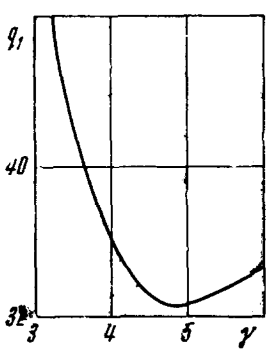

Fig. 2

Therefore, in determining the subcritical state by linear theory the critical load is calculated by the formula

$$Q^* = q^*\mu^3 Eh / 2$$

where $q^* = \text{const} \cong 32.7$. Taking account of the nonlinearity of the subcritical state results in the dependence of q^* on the parameter μ. Numerical values of q^* for a number of values of μ are given below.

μ	0.005	0.010	0.015	0.020	0.025	0.026	0.027
γ^*	4.85	4.8	4.7	4.6	4.4	4.45	4.5
q^*	32.5	31.9	30.9	30.1	29.9	30.1	30.7
ϑ^*	0.162	0.315	0.451	0.572	0.635	0.710	0.740

Values of the parameter γ and absolute values of the subcritical angle of rotation ϑ at the point $x = 0$ ($\vartheta^* = |\vartheta(0)|$ for $q = q^*$) corresponding to q^* are indicated here. For the linear subcritical state $\vartheta^* = \mu q^*$. As we see, taking account of the nonlinearity does not affect the magnitude of the critical load substantially. However, it permits disclosure of that value of μ which limits the domain of existence of the critical load.

The critical load vanished in the problem under consideration between the values $\mu = 0.027$ and $\mu = 0.0275$. Hence, the behavior of the roots of the characteristic determinant $d(q)$ was investigated especially in the interval $0.027 \leqslant \mu \leqslant 0.0275$ for $\gamma^2 = 20$. The results of this investigation are shown in Fig. 3. The solid lines refer to values of μ which differ by 0.0001, where the lowest curve corresponds to the value 0.0270 and the highest to 0.0275. The picture shown indicates that as μ grows, the roots of the determinant (the first and second eigenvalues of the problem) approach each other until they merge for $\mu = \mu_0 \cong 0.0273$. For $\mu > \mu_0$ the determinant has no roots. As μ diminishes from μ_0 the spacing between the roots increases rapidly, as the dashed curve corresponding to the value $\mu = 0.025$ indicates.

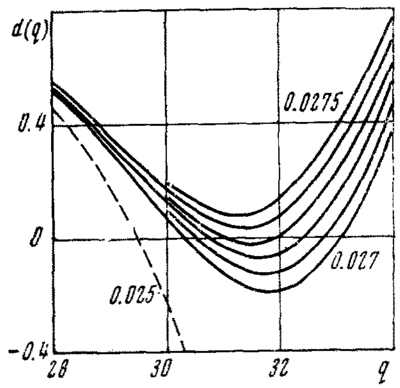

Fig. 3

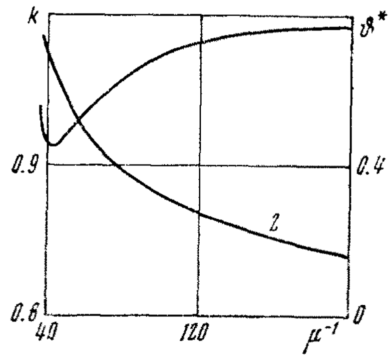

Fig. 4

A graphic representation of the difference between results corresponding to a formulation with the nonlinearity of the subcritical state taken into account (when $q^* = q^*(\mu)$), and a formulation with its linearization ($q^* = q^*(0)$) is given in Fig. 4. The ratio $k = q^*(\mu) / q^*(0)$ is represented here as a function of $\mu^{-1} = [3(1 - \nu^2)]^{1/4} (R/h)^{1/2}$ by curve *1*. The semi-infinite line $k \equiv 1$ corresponds to the solution with linearization of the subcritical state. A still more simplified formulation, when the subcritical curvature of the generator is not generally taken into account, yields $q^* = 8.5$ for $\gamma^* = 1.4$, which agrees with the result in [1]. Curve *2* in Fig. 4 determines the value of ϑ^*

corresponding to the nonlinear problem.

It follows from the results presented that the solution corresponding to the linear subcritical state is asymptotically exact as $\mu \to 0$, but is only suitable for very thin shells ($R / h \geqslant 800$ for $\nu = 0.3$).

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